# TOPOLOGICAL ENTROPY, HOMOLOGICAL GROWTH AND ZETA FUNCTIONS ON GRAPHS

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Abstract. In connection with the Entropy Conjecture it is known that the topological entropy of a continuous graph map is bounded from below by the spectral radius of the induced map on the first homology group. We show that in the case of a piecewise monotone graph map, its topological entropy is equal precisely to the maximum of the mentioned spectral radius and the exponential growth rate of the number of periodic points of negative type. This nontrivially extends a result of Milnor and Thurston on piecewise monotone interval maps. For this purpose we generalize the concept of Milnor-Thurston zeta function incorporating in the Lefschetz zeta function. The methods developed in the paper can be used also in a more general setting.

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## 1. Introduction and the main results

One of the most exciting problems in the theory of dynamical systems for the last three decades has been so called the Entropy Conjecture which was first stated by M. Shub in [S] and which claims that for a smooth map on a compact differentiable manifold, its topological entropy is bounded from below by the logarithm of the spectral radius of the induced map in the corresponding full homology group (called also homological entropy). Many special cases of this conjecture have been already proved even in the nonsmooth case but it still remains open in general. In particular, Manning in [M] proved that the conjecture holds for all continuous maps on differentiable manifolds if we reduce our attention from the full homology group to the first homology group. Using the arguments from the paper it is possible to extend this result for more general spaces including graphs (see [FM]). In this paper we show that for piecewise monotone graph maps, the topological entropy is not only bounded from below by its homological entropy but it is exactly equal to the maximum of its homological entropy and entropy given by the growth of its periodic points of negative type. (In fact, using this result it is possible to provide an alternative proof of the Manning's result for continuous graph maps.)

First we recall some notions and definitions needed in the sequel. By the *homological entropy* of a map $f : X \to X$ we mean a topological invariant $h_{\mathrm{hom}}(f)$ coming from considering the induced linear maps $f_{*i}$ on the homology groups $H_i(X, \mathbb{R})$ and defined by

$$h_{\mathrm{hom}}(f) = \log r(f)$$

where $r(f) = \max\{r(f_{*i}) : i = 0, \dots, \dim X\}$ and $r(f_{*i})$ denotes the spectral radius of $f_{*i}$. The purpose of this paper is to establish a precise relationship between the *topological entropy* $h_{\mathrm{top}}(f)$ and the homological entropy $h_{\mathrm{hom}}(f)$ for a piecewise monotone graph map $f$ (for the definition of topological entropy see any standard textbook on dynamical systems; a nice introduction into the topic can be found in [ALM]). The compact interval and the circle are the simplest examples of graphs. In general, a *graph* is a compact Hausdorff space which can be written as a union of finitely many homeomorphic copies of the closed interval $[0, 1]$ any two of which intersect at most at their endpoints. A point of a graph is called its *vertex* if it does not have any open neighborhood homeomorphic to the open interval $]0, 1[$. The set of all vertices of $G$ is denoted by $\mathrm{Ver}(G)$. Notice

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that if $G$ is a graph and $f : G \to G$ is a continuous map, since $H_i(G, \mathbb{R}) = 0$ for $i \geq 2$, and $r(f_{*0}) = 1$, we obtain

$$r(f) = \max\{1, r(f_{*1})\}.$$

**Definition 1.** Let $G$ be a graph. A continuous map $f : G \to G$ is called a *piecewise monotone graph* (shortly *PMG*) map if there is a finite set $C \subseteq G$ such that $f$ is injective on each connected component of $G \setminus C$.

As mentioned before, our goal is to study the relationship between $h_{\mathrm{top}}(f)$ and $h_{\mathrm{hom}}(f)$. To this end we define another topological invariant $h^-_{\mathrm{per}}(f)$. Let $f : G \to G$ be a PMG map. By $\mathrm{Fix}(f)$ we denote the set of all fixed points of $f$. A point $x \in \mathrm{Fix}(f) \setminus \mathrm{Ver}(G)$ is called *of negative type* if $f$ reverses orientation throughout a small neighborhood of $x$. (Since $x \notin \mathrm{Ver}(G)$, it has a neighborhood homeomorphic to an open real interval on which we can consider $f$ to be a selfmap of the real line.) We denote by $\mathrm{Fix}^-(f)$ the set of all fixed points of negative type of $f$. Evidently, the set $\mathrm{Fix}(f)$ may be infinite but, since $f$ is a PMG map, the set $\mathrm{Fix}^-(f)$ is always finite. Notice that every iterate of a PMG map is again a PMG map. Hence the sets $\mathrm{Fix}^-(f^n)$ are always finite and therefore we can introduce another topological invariant

$$h^-_{\mathrm{per}} = \limsup_{n\to\infty} \frac{1}{n} \log^+ \#\,\mathrm{Fix}^-(f^n),$$

the exponential growth rate of the number of periodic points of negative type (we put $\log^+ x = \log\max\{1, x\}$).

Milnor and Thurston showed in [MT] (see also Theorem 4.11 of [MTr]) that

$$h_{\mathrm{top}}(f) = h^-_{\mathrm{per}}(f) \tag{1}$$

for any piecewise monotone interval map $f$. Among graph maps this does not hold anymore — as an example consider the circle $S^1 = \{x \in \mathbb{C} : |x| = 1\}$ and the map $f : S^1 \to S^1$ defined by $f(x) = x^2$. For this map one obtains $h_{\mathrm{top}}(f) = \log 2$ and $h^-_{\mathrm{per}}(f) = 0$. Nevertheless, we prove the following nice relation extending the last equality.

**Theorem 1.** *Let $f$ be a PMG map. Then*

$$h_{\mathrm{top}}(f) = \max\{h^-_{\mathrm{per}}(f), h_{\mathrm{hom}}(f)\}.$$

The spectral radius $r(f)$ is an algebraic number for any PMG map $f$. Using this we get as a consequence of the last theorem the next result showing that the following entropies are equal for almost all values of topological entropy.

**Corollary 1.** *Let $f$ be a PMG map and suppose that* $\exp(h_{\mathrm{top}}(f))$ *is a transcencental number. Then*

$$h_{\mathrm{top}}(f) = h^-_{\mathrm{per}}(f).$$

Notice that the both results hold for PMG maps in general, even for those with $\mathrm{Fix}(f^n)$ infinite. In the case that $\mathrm{Fix}(f^n)$ is finite for every $n \geq 1$ then we can consider a topological invariant

$$h_{\mathrm{per}}(f) = \limsup_{n\to\infty} \frac{1}{n} \log^+ \#\,\mathrm{Fix}(f^n).$$

For many important cases of PMG maps (expanding maps and more generally maps with “few” stable periodic orbits), topological entropy represents the exponential growth rate for the number of periodic orbits, that is $h_{\mathrm{top}}(f) = h_{\mathrm{per}}(f)$. Moreover, if $\exp(h_{\mathrm{top}}(f))$ is transcendental, we get from the last corollary the following relation

$$h_{\mathrm{top}}(f) = h^-_{\mathrm{per}}(f) = h_{\mathrm{per}}(f). \tag{2}$$

Just stated identity shows that topological entropy in some sense describes the periodic structure of the system in both quantitative and qualitative ways — for an expanding piecewise monotone interval map, we have an obvious relationship between the number of fixed points of negative and positive types (the latter one defined analogously) because between any two consecutive fixed points of $f^n$ of negative type there is exactly one of its fixed point of positive type and consequently

$h^-_{\mathrm{per}}(f) = h_{\mathrm{per}}(f)$. Indeed, we have no such relation between the number of the fixed points of a PMG map of negative and positive types even if the map is expanding.

One of extremely useful tools for studying the relation between topological entropy and the growth of the number of periodic points was introduced by Artin and Mazur in [AM]. Let $X$ be an arbitrary set and $f : X \to X$. The *orbit* of a point $x \in X$ under the action of $f$ is defined as the set $o_x = \{f^n(x) : n \geq 0\}$. An orbit $o_x$ is said to be *periodic* if there is a positive integer $n$ such that $f^n(x) = x$; the smallest such number we denote by $\mathrm{p}(o_x)$ and call its *period*. The set of all periodic orbits of $f$ is denoted by $O$. Suppose that each positive iterate $f^n$ has only finitely many fixed points. Then we define the *Artin-Mazur zeta function* of $f$, $\zeta$, to be the formal power series

$$\zeta(z) = \exp \sum_{n\geq 1} \frac{\#\,\mathrm{Fix}(f^n)}{n} z^n.$$

Recall that the Artin-Mazur zeta function of $f$ is a convenient way of enumerating the periodic orbits of $f$. Indeed, if each positive iterate of $f$ has only finitely many fixed points then the subset $\{o \in O : \mathrm{p}(o) = k\}$ is for any $k$ always finite and the identity

$$\zeta(z) = \prod_{o\in O} \left(1 - z^{\mathrm{p}(o)}\right)^{-1}$$

holds in $\mathbb{Z}[[z]]$, the ring of all formal power series in $z$ over $\mathbb{Z}$.

Later on, several variants of this notion were introduced by different authors (cf. [MT], [BR]; for an extensive survey of the topic see [Ba], [P]; cf. also [R]). In particular, Milnor and Thurston in [MT] modified the Artin-Mazur zeta function to obtain more information for a piecewise monotone interval map. Let us very briefly remind how they arrived at the identity (1). If $f : I \to I$ is a piecewise monotone interval map, we call the formal power series

$$\zeta^{MT}(z) = \exp \sum_{n\geq 1} \frac{2\#\,\mathrm{Fix}^-(f^n) - 1}{n} z^n \tag{3}$$

the *Milnor-Thurston zeta function* of an interval map $f$. Denote its radius of convergence by $\rho$. Starting from the main relation between $\zeta^{MT}(z)$ and the kneading determinant of $f$, Milnor and Thurston proved that

$$h_{\mathrm{top}}(f) = -\log \rho = h^-_{\mathrm{per}}(f).$$

Here we follow the same strategy to prove Theorem 1. As the first step we generalize the concept of Milnor-Thurston zeta function. Let us begin by defining the Lefschetz and negative zeta functions of a PMG map. Let $f : G \to G$ be a PMG map. Recall that the formal power series

$$\zeta^L(z) = \exp \sum_{n\geq 1} \frac{\mathrm{tr}(f_{*0})^n - \mathrm{tr}(f_{*1})^n}{n} z^n$$

is called the *Lefschetz zeta function* of $f$. We define the *negative zeta function* of $f$ as

$$\zeta^-(z) = \exp \sum_{n\geq 1} \frac{2\#\,\mathrm{Fix}^-(f^n)}{n} z^n.$$

Observe that if $f : I \to I$ is a piecewise monotone interval map then we have $\mathrm{tr}(f_{*0}) = 1$ and $\mathrm{tr}(f_{*1}) = 0$ for all $n \geq 1$ and therefore

$$\zeta^-(z)\zeta^L(z)^{-1} = \exp \sum_{n\geq 1} \frac{2\#\,\mathrm{Fix}^-(f^n) - 1}{n} z^n$$

holds in $\mathbb{Z}[[z]]$. So, according to (3) it is natural to define the *Milnor-Thurston zeta function* of a PMG map $f$ as the formal power series

$$\zeta^{MT}(z) = \zeta^-(z)\zeta^L(z)^{-1} \tag{4}$$

and, as before, there is a close relation between $h_{\mathrm{top}}(f)$ and the radius of convergence of $\zeta^{MT}(z)$. Theorem 1 is then an immediate consequence of the following

**Theorem 2.** *Let $f$ be a PMG map and denote by $\rho$ the radius of convergence of $\zeta^{MT}(z)$. Then $0<\rho\leq 1$ and*

$$h_{\mathrm{top}}(f) = -\log\rho.$$

## 2. Proof of Theorem 2

The rest of the paper is devoted to the proof of Theorem 2. In order to simplify notation it is convenient to regard a PMG map $f$ as a real map $F$ with discontinuities defined on a subset of the real line. The proof of Theorem 2 is given in two main steps. The first one is the construction of kneading determinant, $D(z)$, associated to the map $F$. In the second one we set up the relationship between the kneading determinant and the zeta function $\zeta^{MT}(z)$. Because it is not easy to establish a direct relation between $D(z)$ and $\zeta^{L}(z)$, we introduce another determinant, $L(z)$, called the homological determinant of $F$. These two determinants are defined in a very similar way following the techniques introduced in [ASR]. To any $F$ we associate two pairs of linear endomorphisms $(\epsilon F_{\#0},\epsilon F_{\#1})$ and $(F_{\#0},F_{\#1})$. Although these endomorphisms have in general infinite rank, we prove that their difference has always finite rank. This allows us to define $D(z)$ and $L(z)$ as the determinants of these pairs of linear endomorphisms.

This approach is different from the ones used by Milnor and Thurston (see [MT]), Baladi and Ruelle (see [BR]) or Baillif (see [B]). For better readability, we present basic algebraic notions and constructions in the Appendix.

In the remainder of the paper we use the symbol $\Omega$ to denote a finite and disjoint union of compact intervals on the real line

$$\Omega = [a_1,b_1]\cup[a_2,b_2]\cup\cdots\cup[a_m,b_m]$$

with $a_1<b_1<a_2<b_2<\cdots<a_m<b_m$.

**Definition 2.** A *piecewise monotone* (shortly *PM*) map on $\Omega$ is a map $F:\Omega\setminus C_F\to\Omega$ where $C_F$ is a finite subset of $\Omega$ containing $\partial\Omega=\{a_1,b_1,\dots,a_m,b_m\}$ and such that $F$ is continuous and strictly monotone on each connected component of $\Omega\setminus C_F$.

Let $F:\Omega\setminus C_F\to\Omega$ be a PM map and $I=[x,y]$ (with $x<y$) be an interval. We say that $F$ is monotone on $I$ if $]x,y[\subseteq\Omega\setminus C_F$. In this case we define the sign function $\epsilon([x,y])=\pm1$ according to whether $F$ is increasing or decreasing on $]x,y[$. Moreover, for any $x\in\Omega\setminus C_F$, put $\epsilon(x)=\pm1$ according to whether $F$ is increasing or decreasing on a neighborhood of $x$ and put $\epsilon(x)=0$ for every $x\in C_F$. By definition, a *lap* of $F$ is a maximal interval of monotonicity of $F$. That is to say, an interval $I=[c,d]\subseteq\Omega$ (with $c<d$) is a lap of $F$ if and only if $[c,d]\cap C_F=\{c,d\}$. In what follows we use the symbol $\mathcal{L}_F$ to denote the set of all laps of $F$.

For a PM map $F:\Omega\setminus C_F\to\Omega$ and $n$ a positive integer we define its $n$th iterate as a map $F^n:\Omega\setminus C_{F^n}\to\Omega$ inductively by $F^n(x)=F(F^{n-1}(x))$ for all $x\in\Omega\setminus C_{F^n}$ where

$$C_{F^n}=\{x\in\Omega : F^k(x)\in C_F \text{ for some } k=0,\dots,n-1\}.$$

It can be easily seen that this map is PM as well.

Since it is easier to work with PM maps on the real line than with PMG maps we want to replace the latter ones by the first ones. In fact, every PMG map is induced by some PM map on an appropriate set $\Omega$ in the sense of the following

**Definition 3.** Let $f:G\to G$ be a PMG map, $F:\Omega\setminus C_F\to\Omega$ a PM map and $\pi:\Omega\to G$ a continuous map such that $\mathrm{Ver}(G)\subseteq\pi(\partial\Omega)$ and $\pi$ maps $\Omega\setminus\partial\Omega$ homeomorphicaly into $G\setminus\pi(\partial\Omega)$. Then we say that $f$ is *induced* by $F$ if the following diagram

$$\begin{array}{ccc}
\Omega\setminus C_F & \xrightarrow{F} & \Omega \\
\pi\downarrow & & \downarrow\pi \\
G & \xrightarrow{f} & G
\end{array}$$

commutes.

2.1. **The determinants $D(z)$ and $L(z)$.** Let $X$ be a topological space. Denote by $S_0(X;\mathbb{R})$ the $\mathbb{R}$-vector space whose basis consists of the formal symbols $x \in X$, and by $S_1(X)$ its subspace generated by the vectors $y-x$ where $x$ and $y$ are points lying in the same connected component of $X$. If $Y$ is a subset of $X$ and $F : X \setminus Y \to X$ is a map, we denote by $F_{\#0} : S_0(X;\mathbb{R}) \to S_0(X;\mathbb{R})$ the unique linear endomorphism veryfying: $F_{\#0}(x) = F(x)$ if $x \in X \setminus Y$, and $F_{\#0}(x) = 0$ if $x \in Y$.

Let $F : \Omega \setminus C_F \to \Omega$ be a PM map. According to the previous definitions, we have then a vector space $S_0(\Omega;\mathbb{R})$, a subspace $S_1(\Omega;\mathbb{R})$ of $S_0(\Omega;\mathbb{R})$, and a linear endomorphism $F_{\#0} : S_0(\Omega;\mathbb{R}) \to S_0(\Omega;\mathbb{R})$. Notice that both spaces $S_0(\Omega;\mathbb{R})$ and $S_1(\Omega;\mathbb{R})$ are infinite-dimensional but the quotient space $S_0(\Omega;\mathbb{R})/S_1(\Omega;\mathbb{R})$ is finite-dimensional with the dimension equal to the number of connected components of $\Omega$.

Starting from $F_{\#0}$ we define another linear endomorphism $\epsilon F_{\#0} : S_0(\Omega;\mathbb{R}) \to S_0(\Omega;\mathbb{R})$, putting $\epsilon F_{\#0}(x) = \epsilon_F(x) F_{\#0}(x)$ for all $x \in \Omega$. Next we define the linear endomorphisms $F_{\#1} : S_1(\Omega;\mathbb{R}) \to S_1(\Omega;\mathbb{R})$ and $\epsilon F_{\#1} : S_1(\Omega;\mathbb{R}) \to S_1(\Omega;\mathbb{R})$. Notice that, since $F$ is a PM map, the subset of $S_1(\Omega;\mathbb{R})$

$$\mathcal{I}_F = \{y - x : F \text{ is monotone on } [x,y]\}$$

spans $S_1(\Omega;\mathbb{R})$. Furthermore if $F$ is monotone on $[x,y]$ then $F(y-)$ and $F(x+)$ lie in the same connected component of $\Omega$ and therefore $(F(y-) - F(x+)) \in S_1(\Omega;\mathbb{R})$ where $F(y-)$ and $F(x+)$ denote the corresponding one-sided limits. So we can define $F_{\#1}$ and $\epsilon F_{\#1}$ as the unique linear endomorphisms of $S_1(\Omega;\mathbb{R})$ such that

$$F_{\#1}(y-x) = F(y-) - F(x+) \quad \text{and} \quad \epsilon F_{\#1}(y-x) = \epsilon_F([x,y]) F_{\#1}(y-x) \tag{5}$$

for all $y - x \in \mathcal{I}_F$.

As mentioned above, if $F : \Omega \setminus C_F \to \Omega$ is a PM map then $F^n : \Omega \setminus C_{F^n} \to \Omega$ is also a PM map and therefore the linear endomorphisms $F^n_{\#0}$, $F^n_{\#1}$, $\epsilon F^n_{\#0}$ and $\epsilon F^n_{\#1}$ are defined as well. The next lemma is a simple consequence of the definitions and shows that the correspondences $(\cdot)_{\#0}$, $(\cdot)_{\#1}$, $\epsilon(\cdot)_{\#0}$ and $\epsilon(\cdot)_{\#1}$ behave nicely under iteration.

**Lemma 1.** *Let $F : \Omega \setminus C_F \to \Omega$ be a PM map. Then we have $(F_{\#0})^n = F^n_{\#0}$, $(F_{\#1})^n = F^n_{\#1}$, $(\epsilon F_{\#0})^n = \epsilon F^n_{\#0}$ and $(\epsilon F_{\#1})^n = \epsilon F^n_{\#1}$ for all $n \geq 1$.*

Thus for each PM map $F : \Omega \setminus C_F \to \Omega$ we have two pairs of linear endomorphisms on $S_0(\Omega;\mathbb{R})$, $(F_{\#0}, F_{\#1})$ and $(\epsilon F_{\#0}, \epsilon F_{\#1})$. Next we prove that these pairs have both finite ranks (see Definition 4). For this we need first to define extensions of $F_{\#1}$ and $\epsilon F_{\#1}$ to the common superspace $S_0(\Omega;\mathbb{R})$.

For each $c \in [a_i, b_i] \subset \Omega$, let $\alpha^-_c : \Omega \to \mathbb{R}$ and $\alpha^+_c : \Omega \to \mathbb{R}$ be step functions defined by

$$\alpha^-_c(x) = \begin{cases} 1 & \text{for } x \in [c, b_i] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \alpha^+_c(x) = \begin{cases} -1 & \text{for } x \in ]c, b_i] \\ 0 & \text{otherwise.} \end{cases}$$

These step functions induce the linear forms $\omega^-_c : S_0(\Omega;\mathbb{R}) \to \mathbb{R}$ and $\omega^+_c : S_0(\Omega;\mathbb{R}) \to \mathbb{R}$ defined by $\omega^-_c(x) = \alpha^-_c(x)$ and $\omega^+_c(x) = \alpha^+_c(x)$ for all $x \in \Omega$. We also introduce the following notation for special vectors from $S_0(\Omega;\mathbb{R})$: $v^-_c = F(c-)$, $\epsilon v^-_c = \epsilon_F(c-) v^-_c$, $v^+_c = F(c+)$ and $\epsilon v^+_c = \epsilon_F(c+) v^+_c$ putting $v^-_c = 0$ if $c = a_i$, and $v^+_c = 0$ if $c = b_i$.

**Lemma 2.** *Let $F$ be a PM map on $\Omega$. Then the linear endomorphism $\varphi : S_0(\Omega;\mathbb{R}) \to S_0(\Omega;\mathbb{R})$ defined by $\varphi = F_{\#0} + \sum_{c \in C_F} \omega^-_c \otimes v^-_c + \omega^+_c \otimes v^+_c$ is an extension of $F_{\#1}$to $S_0(\Omega;\mathbb{R})$ that verifies $\varphi(S_0(\Omega;\mathbb{R})) \subseteq S_1(\Omega;\mathbb{R})$. Consequently, the pair of linear endomorphisms $(F_{\#0}, F_{\#1})$ has finite rank and*

$$\operatorname{tr}(F_{\#0}, F_{\#1}) = \sum_{c \in C_F} \omega^-_c \left(v^-_c\right) + \omega^+_c \left(v^+_c\right).$$

*Proof.* Let $\varphi : S_0(\Omega;\mathbb{R}) \to S_0(\Omega;\mathbb{R})$ be the linear endomorphism defined by $\varphi(x) = F_{\#1}(x - a_i)$, for all $x \in [a_i, b_i]$. Evidently, $\varphi$ is an extension of $F_{\#1}$to $S_0(\Omega;\mathbb{R})$. Furthermore, since $\varphi(a_i) = 0$ for all $i = 1, \ldots, m$, it follows $\varphi\left(S_0(\Omega;\mathbb{R})\right) \subseteq S_1(\Omega;\mathbb{R})$. On the other hand, from the definitions of

$F_{\#0}$ and $F_{\#1}$, we may write

$$\begin{aligned} F_{\#1}(x-a_i) &= F_{\#0}(x) - F_{\#0}(a_i) + \sum_{c\in ]a_i,x]\cap C_F} v_c^- - \sum_{c\in [a_i,x[\cap C_F} v_c^+ \\ &= F_{\#0}(x) + \sum_{c\in C_F} \omega_c^-(x)\ v_c^- + \omega_c^+(x)\ v_c^+ \end{aligned}$$

for all $x \in [a_i, b_i]$ and thus

$$\varphi(x) = F_{\#0}(x) + \sum_{c\in C_F} \omega_c^-(x)\ v_c^- + \omega_c^+(x)\ v_c^+$$

for all $x \in \Omega$. □

In the same way we can prove the following

**Lemma 3.** *Let $F$ be a PM map on $\Omega$. Then the linear endomorphism $\psi : S_0(\Omega;\mathbb{R}) \to S_0(\Omega;\mathbb{R})$ defined by $\psi = \epsilon F_{\#0} + \sum_{c\in C_F} \epsilon\omega_c^- \otimes v_c^- + \epsilon\omega_c^+ \otimes v_c^+$ is an extension of $\epsilon F_{\#1}$ to $S_0(\Omega;\mathbb{R})$ that verifies $\psi(S_0(\Omega;\mathbb{R})) \subseteq S_1(\Omega;\mathbb{R})$. Consequently, the pair of linear endomorphisms $(\epsilon F_{\#0}, \epsilon F_{\#1})$ has finite rank and*

$$\operatorname{tr}(\epsilon F_{\#0}, \epsilon F_{\#1}) = \sum_{c\in C_F} \omega_c^-(\epsilon v_c^-) + \omega_c^+(\epsilon v_c^+).$$

The last lemma shows that the determinants of the pairs $(\epsilon F_{\#0}, \epsilon F_{\#1})$ and $(F_{\#0}, F_{\#1})$ (see the Appendix) are defined. We define the *kneading determinant* of $F$, $D(z)$, and the *homological determinant* of $F$, $L(z)$, by

$$\begin{aligned} D(z) &= D_{(\epsilon F_{\#0}, \epsilon F_{\#1})}(z) \\ &= \exp - \sum_{n\geq 1} \operatorname{tr}((\epsilon F_{\#0})^n, (\epsilon F_{\#1})^n) \frac{z^n}{n} \end{aligned} \tag{6}$$

and

$$\begin{aligned} L(z) &= D_{(F_{\#0}, F_{\#1})}(z) \\ &= \exp - \sum_{n\geq 1} \operatorname{tr}((F_{\#0})^n, (F_{\#1})^n) \frac{z^n}{n} \end{aligned}$$

Due to Lemmas 2, 3 and Proposition 3 there are vectors $u_1, \dots, u_p \in S_0(\Omega;\mathbb{R})$ and linear forms $\nu_1, \dots, \nu_p$, $\mu_1, \dots, \mu_p \in S_0(\Omega;\mathbb{R})^*$ such that

$$D(z) = \det(\mathbf{Id} - z\mathbf{M}(z)) \text{ and } L(z) = \det(\mathbf{Id} - z\mathbf{N}(z)),$$

where $\mathbf{M}(z) = [\mathbf{m}_{ij}(z)]$ and $\mathbf{N}(z) = [\mathbf{n}_{ij}(z)]$ are $p \times p$ matrices with entries from $\mathbb{Z}[[z]]$ defined by

$$\mathbf{m}_{ij}(z) = \sum_{n\geq 0} \nu_i \circ (\epsilon F_{\#0})^n\,(u_j)\,z^n \quad \text{and} \quad \mathbf{n}_{ij}(z) = \sum_{n\geq 0} \mu_i \circ (F_{\#0})^n\,(u_j)\,z^n. \tag{7}$$

Remark that as a consequence of the definitions the entries of $\mathbf{M}(z)$ and $\mathbf{N}(z)$ are formal power series that can be computed in terms of the orbits of the points of $C_F$ and whose coefficients are from $\{-1, 0, 1\}$. Therefore the entries of $\mathbf{M}(z)$ and $\mathbf{N}(z)$ and the corresponding determinants $D(z)$ and $L(z)$ converge for all $|z| < 1$.

At first glance, it is not clear which kind of relationship can hold between the traces of $(F_{\#0}, F_{\#1})$ and $(\epsilon F_{\#0}, \epsilon F_{\#1})$ and the number of fixed points of $F$. For convenience we introduce the following notation. Let the symbols $\mathcal{L}_F^+$ and $\mathcal{L}_F^-$ denote the set of all laps on which $F$ is increasing and decreasing, respectively. We have then $\mathcal{L}_F = \mathcal{L}_F^- \cup \mathcal{L}_F^+$. For each $I = [c, d] \in \mathcal{L}_F$, define the number

$$\sigma(I) = \omega_c^+\left(v_c^+\right) + \omega_d^-\left(v_d^-\right).$$

Notice that from Lemmas 2 and 3 we have

$$\operatorname{tr}(F_{\#0}, F_{\#1}) = \sum_{I\in\mathcal{L}_F} \sigma(I) \tag{8}$$

and

$$\operatorname{tr}(\epsilon F_{\#0}, \epsilon F_{\#1}) = \sum_{I \in \mathcal{L}_F} \epsilon_F(I)\sigma(I). \tag{9}$$

On the other hand, it is easy to check the following

**Lemma 4.** *Let $F : \Omega \setminus C_F \to \Omega$ be a PM map and $I = [c,d] \in \mathcal{L}_F$. Then $\sigma(I) \in \{-1, 0, 1\}$ and*

$$\sigma(I) = \begin{cases} 1 & \text{if and only if } F(c+) \le c \text{ and } d \le F(d-) \\ -1 & \text{if and only if } c < F(c+) \text{ and } F(d-) < d. \end{cases}$$

We use this result to prove the following main relation between the traces $\operatorname{tr}(\epsilon F_{\#0}, \epsilon F_{\#1})$, and $\operatorname{tr}(F_{\#0}, F_{\#1})$ and the number $\# \operatorname{Fix}^-(F)$.

**Lemma 5.** *Let $F : \Omega \setminus C_F \to \Omega$ be a PM map. Then we have*

$$\operatorname{tr}(\epsilon F_{\#0}, \epsilon F_{\#1}) - \operatorname{tr}(F_{\#0}, F_{\#1}) = 2\# \operatorname{Fix}^-(F).$$

*Proof.* If $I = [c,d] \in \mathcal{L}_F^-$ then there is at most one fixed point of $F$ lying in $]c,d[$ because $F$ is decreasing on $I$, and from Lemma 4 we have: $\sigma(I) = -1$ if there exists a such fixed point; $\sigma(I) = 0$ otherwise. Therefore, from (8) and ( 9), we obtain

$$\begin{aligned} \operatorname{tr}(\epsilon F_{\#0}, \epsilon F_{\#1}) - \operatorname{tr}(F_{\#0}, F_{\#1}) &= -2 \sum_{I \in \mathcal{L}_F^-} \sigma(I) \\ &= 2\# \operatorname{Fix}^-(F) \end{aligned}$$

as desired. □

Notice that by Lemmas 1 and 5 we have

$$\begin{aligned} \operatorname{tr}((\epsilon F_{\#0})^n, (\epsilon F_{\#1})^n) - \operatorname{tr}((F_{\#0})^n, (F_{\#1})^n) &= \operatorname{tr}(\epsilon F^n_{\#0}, \epsilon F^n_{\#1}) - \operatorname{tr}(F^n_{\#0}, F^n_{\#1}) \\ &= 2\# \operatorname{Fix}^-(F^n) \end{aligned}$$

and this proves the main theorem of this subsection,

**Theorem 3.** *Let $F : \Omega \setminus C_F \to \Omega$ be a PM map. Then*

$$\exp \sum_{n \ge 1} \frac{2\# \operatorname{Fix}^-(F^n)}{n} z^n = L(z)D(z)^{-1}$$

*holds in $\mathbb{Z}[[z]]$.*

2.2. **Zeta functions and determinants.** Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. We have then two zeta functions, $\zeta^-(z)$ and $\zeta^L(z)$, and two determinants, $D(z)$ and $L(z)$. By $P$ we denote the union of all periodic orbits of $f$ that intersect $\pi(C_F)$ which is always a finite set. Notice that the numbers $\# \operatorname{Fix}^-(F^n)$ and $\# \operatorname{Fix}^-(f^n)$ do not need to coincide because there may exist periodic orbits of $f$ which intersect simultaneously $\pi(C_F)$ and $\# \operatorname{Fix}^-(f^n)$ for some $n \ge 1$. Nevertheless we have

$$\# \operatorname{Fix}^-(f^n) - \# \operatorname{Fix}^-(F^n) = \#P \cap \operatorname{Fix}^-(f^n) \tag{10}$$

for all $n \ge 1$, and consequently

$$\max\left\{1, \limsup_{n\to\infty} \# \operatorname{Fix}^-(F^n)^{1/n}\right\} = \max\left\{1, \limsup_{n\to\infty} \# \operatorname{Fix}^-(f^n)^{1/n}\right\} \tag{11}$$

As an immediate consequence of (10) and Theorem 3 we also have:

**Corollary 2.** *Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. Then*

$$\zeta^-(z) = L(z)D(z)^{-1} \exp \sum_{n \ge 1} \frac{2\#P \cap \operatorname{Fix}^-(f^n)}{n} z^n$$

*holds in $\mathbb{Z}[[z]]$.*

The next result, together with Corollary 2, allow us to establish a main relationship between $\zeta^{MT}(z)$ and $D(z)$.

**Theorem 4.** *Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. Then*

$$\zeta^L(z) = L(z) \exp \sum_{n\geq 1} \frac{\#P \cap \mathrm{Fix}(f^n)}{n} z^n$$

*holds in $\mathbb{Z}[[z]]$.*

In order to prove Theorem 4, let us begin by defining an auxiliary pair of linear endomorphisms on $S_0(G;\mathbb{R})$. Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$ and consider the linear endomorphisms $\beta_0 : S_0(G;\mathbb{R}) \to S_0(G;\mathbb{R})$ and $\beta_1 : S_1(G;\mathbb{R}) \to S_1(G;\mathbb{R})$ defined by $\beta_0(x) = \begin{cases} f(x) & \text{if } x \in G \setminus \pi(C_F) \\ 0 & \text{if } x \in \pi(C_F) \end{cases}$ and $\beta_1(y - x) = f(y) - f(x)$ for all $x$ and $y$ lying in the same connected component of $G$. The next lemma shows that $(\beta_0, \beta_1)$ has finite rank and relates the determinants $D_{(\beta_0,\beta_1)}(z)$ and $D_{f_{*0}}(z)$.

**Lemma 6.** *Under the conditions of the previous theorem, the pair $(\beta_0, \beta_1)$ of linear endomorphisms on $S_0(G;\mathbb{R})$ has finite rank and*

$$D_{(\beta_0,\beta_1)}(z) D_{f_{*0}}(z) = \exp \sum_{n\geq 1} -\frac{\#P \cap \mathrm{Fix}(f^n)}{n} z^n$$

*holds in $\mathbb{Z}[[z]]$.*

*Proof.* Let $\beta : S_0(G;\mathbb{R}) \to S_0(G;\mathbb{R})$ be the unique linear endomorphism that verifies $\beta(x) = f(x)$ for all $x \in G$. Evidently, $\beta$ is an extension of $\beta_1$ to $S_0(G;\mathbb{R})$. Since $\beta(x) = \beta_0(x)$ for all $x \in G \setminus \pi(C_F)$ and $\pi(C_F)$ is a finite set, we see that $\beta - \beta_0$ has finite rank. This shows that $(\beta_0, \beta_1)$ has finite rank. On the other hand, because

$$S_0(G;\mathbb{R}) / S_1(G;\mathbb{R}) = H_0(G;\mathbb{R}),$$

we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_1(G;\mathbb{R}) & \longrightarrow & S_0(G;\mathbb{R}) & \xrightarrow{\pi_0} & H_0(G;\mathbb{R}) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \beta_1} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle f_{*0}} & & \\ 0 & \longrightarrow & S_1(G;\mathbb{R}) & \longrightarrow & S_0(G;\mathbb{R}) & \xrightarrow{\pi_0} & H_0(G;\mathbb{R}) & \longrightarrow & 0 \end{array}$$

Thus from Definition 4 we have

$$\mathrm{tr}(\beta_0^n, \beta_1^n) + \mathrm{tr}(f_{*0}^n) = \mathrm{tr}(\beta^n - \beta_0^n)$$

for all $n \geq 1$, and the proof follows because as an immediate consequence of the definitions one has $\mathrm{tr}(\beta^n - \beta_0^n) = \#P \cap \mathrm{Fix}(f^n)$ for all $n \geq 1$. $\square$

Let us start now to prove Theorem 4. Notice that if $f : G \to G$ is a PMG map induced by $F : \Omega \setminus C_F \to \Omega$ then the continuous map $\pi : \Omega \to G$ induces linear endomorphisms

$$\pi_0 : S_0(\Omega;\mathbb{R}) \to S_0(G;\mathbb{R}) \qquad \text{and} \qquad \pi_1 : S_1(\Omega;\mathbb{R}) \to S_1(G;\mathbb{R})$$

where $\pi_0$ is the unique linear map that verifies $\pi_0(x) = \pi(x)$ for all $x \in \Omega$, and $\pi_1$ is the restriction of $\pi_0$ to $S_1(\Omega;\mathbb{R})$ (since $\pi$ is a continuous map, $\pi_0$ maps $S_1(\Omega;\mathbb{R})$ in to $S_1(G;\mathbb{R})$). Since $\mathrm{Ker}(\pi_1) = \mathrm{Ker}(\pi_0) \cap S_1(\Omega;\mathbb{R}) \subset \mathrm{Ker}(\pi_0)$, we can consider the pair $(\alpha_0, \alpha_1)$ of linear endomorphisms on $\mathrm{Ker}(\pi_0)$, where $\alpha_i$ is the restriction of $F_{\#i}$ to $\mathrm{Ker}(\pi_i)$. Notice that, since $\pi$ maps $\Omega \setminus \partial\Omega$ homeomorphicaly into $G \setminus \pi(\partial\Omega)$, we have $\mathrm{Ker}(\pi_0) \subset S_0(\partial\Omega;\mathbb{R}) \subset \mathrm{Ker}(F_{\#0})$, and thus $\alpha_0 = 0$. On the other hand we also have $H_1(G;\mathbb{R}) = \mathrm{Ker}(\pi_1)$ and $\alpha_1 = f_{*1}$. This shows that the pair $(\alpha_0, \alpha_1)$ has finite rank and

$$D_{(\alpha_0,\alpha_1)}(z) = D_{(0,f_{*1})}(z) = D_{f_{*1}}(z) \tag{12}$$

Thus we obtain three pairs of linear endomorphisms with finite rank, $(\alpha_0, \alpha_1)$, $(F_{\#0}, F_{\#1})$ and $(\beta_0, \beta_1)$ on $\operatorname{Ker}(\pi_0)$, $S_0(\Omega; \mathbb{R})$ and $S_0(G; \mathbb{R})$, respectively, and the two following commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker}\pi_0 & \longrightarrow & S_0(\Omega;\mathbb{R}) & \xrightarrow{\pi_0} & S_0(G;\mathbb{R}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle F_{\#0}} & & \downarrow{\scriptstyle f_{\#0}} & & \\ 0 & \longrightarrow & \operatorname{Ker}\pi_0 & \longrightarrow & S_0(\Omega;\mathbb{R}) & \xrightarrow{\pi_0} & S_0(G;\mathbb{R}) & \longrightarrow & 0 \end{array} \tag{13}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker}\pi_1 & \longrightarrow & S_1(\Omega;\mathbb{R}) & \xrightarrow{\pi_1} & S_1(G;\mathbb{R}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle F_{\#1}} & & \downarrow{\scriptstyle \beta_1} & & \\ 0 & \longrightarrow & \operatorname{Ker}\pi_1 & \longrightarrow & S_1(\Omega;\mathbb{R}) & \xrightarrow{\pi_1} & S_1(G;\mathbb{R}) & \longrightarrow & 0 \end{array} \tag{14}$$

The restriction of $\pi_0$ to $S_1(\Omega; \mathbb{R})$ is $\pi_1$, so from Proposition 4 and Lemma 6 we obtain

$$\begin{aligned} L(z) &= D_{(F_{\#0},F_{\#1})}(z) \\ &= D_{(\alpha_0,\alpha_1)}(z) D_{(\beta_0,\beta_1)}(z) \\ &= D_{(\alpha_0,\alpha_1)}(z) D_{f_{*0}}(z)^{-1} \exp \sum_{n\geq 1} -\frac{\#P \cap \operatorname{Fix}(f^n)}{n} z^n \end{aligned}$$

and from (5)

$$\begin{aligned} L(z) &= D_{f_{*1}}(z) D_{f_{*0}}(z)^{-1} \exp \sum_{n\geq 1} -\frac{\#P \cap \operatorname{Fix}(f^n)}{n} z^n \\ &= \zeta^L(z) \exp \sum_{n\geq 1} -\frac{\#P \cap \operatorname{Fix}(f^n)}{n} z^n \end{aligned}$$

as desired.

From Corollary 2 and Theorem 4 we obtain

**Theorem 5.** *Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. Then there exists a formal power series $H(z)$ such that $H(z)$ converges and is nonzero for all $|z| < 1$ and*

$$\zeta^{MT}(z) = H(z) D(z)^{-1}$$

*holds in $\mathbb{Z}[[z]]$.*

*Proof.* We have

$$\zeta^{MT}(z) = \zeta^-(z)\zeta^L(z)^{-1} = D(z)^{-1} \exp a(z)$$

with

$$a(z) = \sum_{n\geq 1} \frac{2\#P \cap \operatorname{Fix}^-(f^n) - \#P \cap \operatorname{Fix}(f^n)}{n} z^n.$$

Thus, because $P$ is a finite set, it follows immediately that $a(z)$ converges for all $|z| < 1$ and consequently $H(z) = \exp a(z) \neq 0$ for all $|z| < 1$. □

As mentioned before, the kneading determinant $D(t)$ converges for all $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, so, as immediate consequence of Theorem 5, we obtain

**Corollary 3.** *Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$, $\rho$ the radius of convergence of $\zeta^{MT}(z)$, and $z_0$ a zero of $D(z)$ lying in $\mathbb{D}$. Then we have $\rho \leq |z_0|$.*

Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. Next we discuss the relationship between topological entropy of $f$ and the zeros of $D(z)$. From now on we use the symbol $\ell(F^n)$ to denote the number of laps of the iterate $F^n : \Omega \setminus C_{F^n} \to \Omega$, in other words $\ell(F^n) = \#\mathcal{L}(F^n)$.

If $I=[c,d]\in\mathcal{L}(F^n)$ we define the variation of $F^n$ on $I$ by $\mathrm{Var}_i(F^n)=|F^n(d-)-F^n(d+)|$. The variation of $F^n$ is defined by

$$\mathrm{Var}(F^n)=\sum_{I\in\mathcal{L}(F^n)}\mathrm{Var}_i(F^n).$$

Recall that, for interval and circle maps Misiurewicz and Szlenk proved in [MSz] that

$$h_{\mathrm{top}}(f)=\log\lim_{n\to\infty}\ell(F^n)^{1/n}=\log\max\left\{1,\lim_{n\to\infty}\mathrm{Var}(F^n)^{1/n}\right\},\tag{15}$$

and the same arguments can be adapted to show that (15) holds for any PMG map.

Let us begin with the following

**Theorem 6.** *Let $f:G\to G$ be a PMG map induced by $F:\Omega\setminus C_F\to\Omega$. If $h_{\mathrm{top}}(f)>0$. Then $D(z)=0$ for some $|z|=\lim_{n\to\infty}\mathrm{Var}(F^n)^{-1/n}$.*

*Proof.* Let $v=(b_1-a_1)+...+(b_m-a_m)\in S_1(\Omega;\mathbb{R})$ and $\xi:S_1(\Omega;\mathbb{R})\to\mathbb{R}$ the linear form defined by $\xi(y-x)=y-x$, for all $x$ and $y$ lying in the same connected component of $\Omega$. We have then a pair $(\epsilon F_{\#0},\epsilon F_{\#1}+\xi\otimes v)$ of endomorphisms on $S_1(\Omega;\mathbb{R})$ with finite rank. As mentioned before, the kneading determinant $D(z)=D_{(\epsilon F_{\#0},\epsilon F_{\#1})}(z)$ converges for all $|z|<1$. Using the same argument, it is easy to show that $D_{(\epsilon F_{\#0},\epsilon F_{\#1}+\xi\otimes v)}(z)$ also converges for all $|z|<1$.

Notice that if $I=[c,d]$ is a lap of $F^n$, we have $\mathrm{Var}_i(F^n)=\xi\circ\epsilon F^n_{\#1}(d-c)$. Thus, from the linearity of $\epsilon F^n_{\#1}$, and by Lemma 1, we arrive at

$$\mathrm{Var}(F^n)=\xi\circ\epsilon F^n_{\#1}(v)=\xi\circ\left(\epsilon F_{\#1}\right)^n(v),$$

since $h_{\mathrm{top}}(f)>0$, we have then

$$\limsup_{n\to\infty}\left|\xi\circ\left(\epsilon F_{\#1}\right)^n(v)\right|^{1/n}=\lim_{n\to\infty}\mathrm{Var}(F^n)^{1/n}>1,$$

and from Proposition 5, $D(z)=0$, for some $|z|=\lim_{n\to\infty}\mathrm{Var}(F^n)^{-1/n}$. □

We have now everything we needed to prove Theorem 2. Let $f:G\to G$ be a PMG map induced by $F:\Omega\setminus C_F\to\Omega$, and denote by $\rho$ the fradius of convergence of $\zeta_F^{MT}(z)$. From the definition of $\zeta_F^{MT}(z)$, we see at once that

$$\rho^{-1}\le\max\left\{1,\limsup_{n\to\infty}\#\mathrm{Fix}^-(f^n)^{1/n},r(f_{*1})\right\},$$

and thus

$$\log\max\left\{1,\rho^{-1}\right\}\le\max\left\{h^-_{\mathrm{per}}(f),h_{\mathrm{hom}}(f)\right\}.$$

Suppose that $h_{\mathrm{top}}(f)>0$. In this case, from (15) and Theorem 6, we have $D(z)=0$, for some $|z|=\lim_{n\to\infty}\mathrm{Var}(F^n)^{-1/n}<1$. Thus from Corollary 3, we have $\rho\le\lim_{n\to\infty}\mathrm{Var}(F^n)^{-1/n}$ and thus

$$h_{\mathrm{top}}(f)\le\log\max\left\{1,\rho^{-1}\right\}.$$

Finally it remains to prove $\max\left\{h^-_{\mathrm{per}}(f),h_{\mathrm{hom}}(f)\right\}\le h_{\mathrm{top}}(f)$. Since in each lap of $F^n$ there is at most one fixed point of negative type, we have

$$\lim_{n\to\infty}\ell\left(F^n\right)^{1/n}\ge\max\left\{1,\limsup_{n\to\infty}\#\,\mathrm{Fix}^-(F^n)^{1/n}\right\},$$

and from (11)

$$\lim_{n\to\infty}\ell\left(F^n\right)^{1/n}\ge\max\left\{1,\limsup_{n\to\infty}\#\,\mathrm{Fix}^-(f^n)^{1/n}\right\}.$$

Thus, by (15), $h_{\mathrm{top}}(f)\ge h^-_{\mathrm{per}}(f)$. From (8) and Lemma 1 we also have

$$\begin{aligned}\ell\left(F^n\right)&\ge\left|\sum_{I\in\mathcal{L}_{F^n}}\sigma(I)\right|\\&=\left|\mathrm{tr}(F^n_{\#0},F^n_{\#1})\right|\\&=\left|\mathrm{tr}(\left(F_{\#0}\right)^n,\left(F_{\#1}\right)^n)\right|\end{aligned}$$

for all $n \geq 1$. But from Theorem 4

$$|\operatorname{tr}((F_{\#0})^n, (F_{\#1})^n)| = |\#P \cap \operatorname{Fix}(f^n) + \operatorname{tr}(f_{*1})^n - \operatorname{tr}(f_{*0})^n|$$

for all $n \geq 1$. Thus, since $P$ is a finite set, it follows

$$\lim_{n\to\infty} \ell\,(F^n)^{1/n} \geq \max\left\{1, \limsup_{n\to\infty} |\operatorname{tr}(f_{*1})^n|^{1/n}\right\} = r(f),$$

and, once again from (15), $h_{\text{top}}(f) \geq h_{\text{hom}}(f)$.

## Appendix (Pairs of linear endomorphisms)

Let $V$ be a vector space over $\mathbb{R}$ and let $\varphi : V \to V$ be a linear map with finite rank. As usually we define the trace of $\varphi$ by

$$\operatorname{tr}\varphi = \operatorname{tr}\varphi_{|\operatorname{Im}\varphi}.$$

If $\varphi$ has finite rank then there are vectors $v_1, \dots, v_k \in V$ and linear forms $\omega_1, \dots, \omega_k \in V^*$ such that

$$\varphi = \sum_{i=1}^{k} \omega_i \otimes v_i.$$

Considering the matrix

$$\mathbf{M} = \begin{pmatrix} \omega_1(v_1) & \dots & \omega_1(v_k) \\ \vdots & & \vdots \\ \omega_k(v_1) & \dots & \omega_k(v_k) \end{pmatrix} \tag{16}$$

we have

$$\operatorname{tr}\varphi = \operatorname{tr}\mathbf{M}.$$

More generally, if $\varphi$ has finite rank then $\varphi^n$, $n \geq 1$, has also finite rank and

$$\operatorname{tr}\varphi^n = \operatorname{tr}\mathbf{M}^n.$$

The following proposition is well known and gives an explicit method for computing the numbers $\operatorname{tr}\varphi^n$ for $n \geq 1$. Defining the determinant of $\varphi$ to be the following formal power series

$$D_\varphi(z) = \exp \sum_{n\geq 1} -\operatorname{tr}\varphi^n \frac{z^n}{n}$$

we have

**Proposition 1.** *Let $\varphi$ be an endomorphism with finite rank. Then we have*

$$D_\varphi(z) = \det(\mathbf{Id} - z\mathbf{M}) \qquad \textit{in } \mathbb{R}[[z]].$$

Now we consider a more general situation. By a pair of endomorphisms $(\varphi_0, \varphi_1)$ on $V$ we mean two two linear maps $\varphi_0 : V_0 \to V_0$ and $\varphi_1 : V_1 \to V_1$ defined on two finite-codimensional subspaces $V_0$ and $V_1$ of the same $\mathbb{R}$-vector space $V$.

**Definition 4.** We say that the pair of endomorphisms $(\varphi_0, \varphi_1)$ on $V$ has a *finite rank* if there exist linear maps $\overline{\varphi}_0$, $\widetilde{\varphi}_0$, $\overline{\varphi}_1$ and $\widetilde{\varphi}_1$ such that the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_j & \xrightarrow{\subseteq} & V & \xrightarrow{\text{pr}} & V/V_j & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \varphi_j} & & \Big\downarrow{\scriptstyle \overline{\varphi}_j} & & \Big\downarrow{\scriptstyle \widetilde{\varphi}_j} & & \\ 0 & \longrightarrow & V_j & \xrightarrow{\subseteq} & V & \xrightarrow{\text{pr}} & V/V_j & \longrightarrow & 0 \end{array}$$

commutes for $j = 0, 1$ and the linear map $\overline{\varphi}_1 - \overline{\varphi}_0$ has finite rank. The *trace* of a pair $(\varphi_0, \varphi_1)$ with finite rank is defined by $\operatorname{tr}(\varphi_0, \varphi_1) = \operatorname{tr}(\overline{\varphi}_1 - \overline{\varphi}_0) - \operatorname{tr}\widetilde{\varphi}_1 + \operatorname{tr}\widetilde{\varphi}_0$.

It is easy to see that the definition does not depend on $\overline{\varphi}_0$, $\widetilde{\varphi}_0$, $\overline{\varphi}_1$ and $\widetilde{\varphi}_1$. As an immediate consequence of the definition we get

**Proposition 2.** *Let $(\varphi_0, \varphi_1)$ be a pair of endomorphisms on $V$, and $\overline{\varphi}_i : V \to V$ an extension of $\varphi_i$ such that $\overline{\varphi}_i(V) \subseteq V_i$, for $i = 0, 1$. Then $(\varphi_0, \varphi_1)$ has finite rank if and only if $\overline{\varphi}_1 - \overline{\varphi}_0$ has finite rank. Furthermore, if $(\varphi_0, \varphi_1)$ has finite rank then* $\operatorname{tr}(\varphi_0, \varphi_1) = \operatorname{tr}(\overline{\varphi}_1 - \overline{\varphi}_0)$.

Let $(\varphi_0, \varphi_1)$ be a pair of endomorphisms on $V$ having finite rank, and consider endomorphisms $\overline{\varphi}_0$ and $\overline{\varphi}_1$ as in Proposition 2. Since $\overline{\varphi}_1 - \overline{\varphi}_0$ has finite rank, we can consider vectors $v_1, \dots, v_k \in V$ and linear forms $\omega_1, \dots, \omega_k \in V^*$ such that

$$\overline{\varphi}_1 - \overline{\varphi}_0 = \sum_{i=1}^{k} \omega_i \otimes v_i. \tag{17}$$

More generally, we have

$$\overline{\varphi}_1^n - \overline{\varphi}_0^n = \sum_{i=1}^{k} \sum_{j=1}^{n} \left( \omega_i \circ \overline{\varphi}_1^{n-j} \right) \otimes \overline{\varphi}_0^{j-1}(v_i),$$

for each $n \geq 1$. This shows that $\overline{\varphi}_1^n - \overline{\varphi}_0^n$ has finite rank for each $n \geq 1$. Thus, once more from Proposition 2, we conclude that the pair $(\varphi_0^n, \varphi_1^n)$ has finite rank and

$$\operatorname{tr}(\varphi_0^n, \varphi_1^n) = \operatorname{tr}(\overline{\varphi}_1^n - \overline{\varphi}_0^n) \qquad \text{for each } n \geq 1.$$

**Definition 5.** Let $(\varphi_0, \varphi_1)$ be a pair of endomorphisms having finite rank. We define the *determinant* of $(\varphi_0, \varphi_1)$ to be the following element of $\mathbb{R}[[z]]$

$$D_{(\varphi_0, \varphi_1)}(z) = -\sum_{n \geq 1} \operatorname{tr}(\varphi_0^n, \varphi_1^n) \frac{z^n}{n}.$$

Observe that if $\varphi$ has finite rank then

$$D_{(0,\varphi)}(z) = D_{\varphi}(z).$$

If $\varphi_0$ and $\varphi_1$ both have finite ranks then

$$D_{(\varphi_0, \varphi_1)}(z) = D_{\varphi_1}(z) D_{\varphi_0}(z)^{-1}.$$

So, in these cases, we can use Proposition 1 for computing $D_{(\varphi_0, \varphi_1)}(z)$. Obviously, in the general case, Proposition 1 does not allow us to compute $D_{(\varphi_0, \varphi_1)}(z)$ — $D_{\varphi_0}(z)$ and $D_{\varphi_1}(z)$ are not defined in general.

In order to compute $D_{(\varphi_0, \varphi_1)}(z)$ in the general case, we generalize the Proposition 1. Let $\overline{\varphi}_0$ and $\overline{\varphi}_1$ be endomorphisms as in Proposition 2. Considering vectors $v_1, \dots, v_k \in V$ and linear forms $\omega_1, \dots, \omega_k \in V^*$ as in (17), we define the matrix

$$\mathbf{M}(z) = \begin{pmatrix} \sum_{n \geq 0} \omega_1(\overline{\varphi}_0^n(v_1)) z^n & \dots & \sum_{n \geq 0} \omega_1(\overline{\varphi}_0^n(v_k) z^n \\ \vdots & & \vdots \\ \sum_{n \geq 0} \omega_k(\overline{\varphi}_0^n(v_1)) z^n & \dots & \sum_{n \geq 0} \omega_k(\overline{\varphi}_0^n(v_k) z^n \end{pmatrix} \tag{18}$$

with coefficients in $\mathbb{R}[[z]]$. Observe that if we identify an endomorphism with finite rank $\varphi : V \to V$ with the corresponding pair of finite rank $(0, \varphi)$ then the matrix $\mathbf{M}(z)$ from (18 ) coincides with the matrix $\mathbf{M}$ defined in (16). Thus the next proposition which gives an explicit method for computing $D_{(\varphi, \psi)}(z)$, can be regarded as a generalization of Proposition 1.

**Proposition 3.** *Let $(\varphi_0, \varphi_1)$ be a pair of endomorphisms having finite rank. Then*

$$D_{(\varphi_0, \varphi_1)}(z) = \det(\mathbf{Id} - z\mathbf{M}(z)).$$

Let $0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$ be an exact sequence of $\mathbb{R}$-vector spaces. Recall that if the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{i} & V & \xrightarrow{p} & W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & U & \xrightarrow{i} & V & \xrightarrow{p} & W & \longrightarrow & 0 \end{array}$$

commutes and the endomorphisms $\chi$, $\varphi$ and $\psi$ have finite rank then we have

$$\operatorname{tr}\varphi^n = \operatorname{tr}\chi^n + \operatorname{tr}\psi^n$$

for all $n \geq 1$, and therefore

$$D_\varphi(z) = D_\chi(z)D_\psi(z).$$

The next proposition can be regarded as a generalization of the last formula. Let $(\chi_0, \chi_1)$, $(\varphi_0, \varphi_1)$ and $(\psi_0, \psi_1)$ be pairs of endomorphisms in $U$, $V$ and $W$ respectively, such that the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_j & \xrightarrow{i} & V_j & \xrightarrow{p} & W_j & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \chi_j} & & \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle \psi_j} & & \\
0 & \longrightarrow & U_j & \xrightarrow{i} & V_j & \xrightarrow{p} & W_j & \longrightarrow & 0
\end{array}$$

commutes for $j = 0, 1$. Then we have the following

**Proposition 4.** *Let $(\chi_0, \chi_1)$, $(\varphi_0, \varphi_1)$ and $(\psi_0, \psi_1)$ be pairs of endomorphisms with finite rank such that the last diagram above commutes. Then*

$$\operatorname{tr}(\varphi_0^n, \varphi_1^n) = \operatorname{tr}(\chi_0^n, \chi_1^n) + \operatorname{tr}(\psi_0^n, \psi_1^n) \qquad \textit{for all } n \geq 1, \textit{ and}$$
$$D_{(\varphi_0,\varphi_1)}(z) = D_{(\chi_0,\chi_1)}(z) D_{(\psi_0,\psi_1)}(z).$$

Let us consider, for the last time, a linear endomorphism $\varphi : V \to V$ with finite rank. Recall that, if $v \in V$ and $\xi \in V^*$, then there exists an eigenvalue, $\lambda$, of $\varphi$ such that

$$\limsup_{n\to\infty} |\xi \circ \varphi^n(v)|^{1/n} = |\lambda|$$

and consequently

$$D_\varphi(z) = 0 \text{ for some } |z| = \frac{1}{\limsup_{n\to\infty} |\xi \circ \varphi^n(v)|^{1/n}}. \tag{19}$$

We will finish this appendix with a generalization of (19). Let $(\varphi_0, \varphi_1)$ be a pair of endomorphisms on $V$ with finite rank, $v \in V_1$, $\xi \in V_1^*$. Then the pair $(\varphi_0, \varphi_1 + \xi \otimes v)$ of endomorphisms on $V$, also has finite rank. Notice that, since $D_{(\varphi_0,\varphi_1)}(z)$ and $D_{(\varphi_0,\varphi_1+\xi\otimes v)}(z)$ are not (in general) polynomials, we have to assumme that there exists $r > 0$ such that $D_{(\varphi_0,\varphi_1)}(z)$ and $D_{(\varphi_0,\varphi_1+\xi\otimes v)}(z)$ converge for all $|z| < r$. If we consider the pair $(\varphi_1, \varphi_1 + \xi \otimes v)$ of endomorphisms on $V_1$, this pair has evidently finite rank, and from Proposition 3 we see that

$$D_{(\varphi_1,\varphi_1+\xi\otimes v)}(z) = 1 - \sum_{n\geq 0} \xi \circ \varphi_1^n(v) z^{n+1}$$

holds in $\mathbb{R}[[z]]$. On the other hand, regarding $(\varphi_1, \varphi_1 + \xi \otimes v)$ as a a pair of endomorphisms on $V$, we have the decomposition

$$D_{(\varphi_1,\varphi_1+\xi\otimes v)}(z) = D_{(\varphi_1,\varphi_0)}(z) D_{(\varphi_0,\varphi_1+\xi\otimes v)}(z) = \frac{D_{(\varphi_0,\varphi_1+\xi\otimes v)}(z)}{D_{(\varphi_0,\varphi_1)}(z)},$$

and therefore

$$1 - \sum_{n\geq 0} \xi \circ \varphi_1^n(v) z^{n+1} = \frac{D_{(\varphi_0,\varphi_1+\xi\otimes v)}(z)}{D_{(\varphi_0,\varphi_1)}(z)}$$

holds in in $\mathbb{R}[[z]]$. Thus, since the radius of convergence of

$$1 - \sum_{n\geq 0} \xi \circ \varphi_1^n(v) z^{n+1}$$

is

$$\rho = \frac{1}{\limsup_{n\to\infty} |\xi \circ \varphi_1^n(v)|^{1/n}},$$

and the function

$$\gamma(z) = \frac{D_{(\varphi_0,\varphi_1+\xi\otimes v)}(z)}{D_{(\varphi_0,\varphi_1)}(z)}$$

is meromorphic on $|z| < r$, we can conclude: if $\rho < r$ then $\gamma(z)$ has a pole lying in $|z| = \rho$. So, because the poles of $\gamma(z)$ are zeros of $D_{(\varphi_0,\varphi_1)}(z)$, we may write:

**Proposition 5.** *Let $(\varphi_0, \varphi_1)$ be a pair of endomorphisms on $V$ with finite rank, $v \in V_1$, $\xi \in V_1^*$ and $r > 0$ such that $D_{(\varphi_0,\varphi_1)}(z)$ and $D_{(\varphi_0,\varphi_1+\xi\otimes v)}(z)$ converge for all $|z| < r$, and $\limsup_{n\to\infty} |\xi \circ \varphi_1^n(v)|^{1/n} > r^{-1}$. Then we have*

$$D_{(\varphi_0,\varphi_1)}(z) = 0, \text{ for some } |z| = \frac{1}{\limsup_{n\to\infty} |\xi \circ \varphi_1^n(v)|^{1/n}}$$

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